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# The spin contribution to the form factor of quantum graphs

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## Abstract

Following the quantisation of a graph with the Dirac operator (spin-1/2) we explain how additional weights in the spectral form factor  $K(\tau)$  due to spin propagation around orbits produce higher order terms in the small- $\tau$  asymptotics in agreement with symplectic random matrix ensembles. We determine conditions on the group of spin rotations sufficient to generate CSE statistics.

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# 1 Introduction

Overwhelming evidence shows that correlations in discrete energy spectra of classically chaotic quantum systems generally follow the conjecture of Bohigas, Giannoni, and Schmit [4]. According to this the spectral statistics can be described by random matrix theory (RMT), the universality classes being completely determined by symmetry. Berry [3] analysed spectral two-point correlations measured with the form factor  $K(\tau)$  obtaining the leading order in its small- $\tau$  asymptotics in agreement with RMT. Recent developments suggest further asymptotic terms can be attributed to correlations between classical periodic orbits with (almost) self-intersections [12, 11]. Implicit to this method is the assumption that the  $\tau^m$ -term in the form factor expansion is determined only by correlations between pairs of orbits where the order of sections of the orbit has been changed at  $m - 1$  (almost) self-intersections. The original results of Sieber and Richter, concerning systems on surfaces of constant negative curvature, were extended to a class of graphs quantised with the Schrödinger operator by Berkolaiko, Schanz, and Whitney [2, 1]. They find agreement with the form factor  $K_{\text{COE}}(\tau)$  of the circular orthogonal RMT-ensemble (COE) to third order in  $\tau$  by considering correlations between orbits with up to two self-intersections.

In [5] we showed that a graph quantised with the Dirac operator and possessing time-reversal symmetry produces level statistics which agree numerically with those of the Gaussian symplectic ensemble (GSE), in accordance with [4]. The form factor differs from that of the usual Schrödinger quantisation via weights determined by the spin transformations around classical periodic orbits. In the diagonal approximation we found an agreement of the form factor with  $K_{\text{GSE}}(\tau)$  in first order. Here we show that the spin weights generate the additional pre-factors relating the expansions of  $K_{\text{CSE}}(\tau)$  and  $K_{\text{COE}}(\tau)$  in all orders. Our approach avoids the various ad hoc assumptions introduced in the related investigation by Heusler [9] and allows us to state precise conditions on the spin dynamics. The result also provides strong evidence for the hypothesis that the  $\tau^m$ -term derives only from pairs of orbits differing in the order of sections at  $m - 1$  self-intersections.

## 2 The form factor

In our previous work [5] we studied correlations in spectra of Dirac operators on graphs. Here we rather adopt the closely related point of view taken in [2, 1] and consider the form factor derived from the spectrum of the  $S$ -matrix that was introduced in [5]. On the side of RMT we hence have to consider the circular instead of the Gaussian ensembles.

We recall that a (compact) graph consists of  $V$  vertices connected by  $B$  bonds. The valency of a vertex  $i$  is  $v_i$ . Let  $(ij)$  label a transition from vertex  $i$  to vertex  $j$  along a bond  $\{ij\}$ . According to [5] for fixed energy there exist two linearly independent eigenspinors of the Dirac operator for each transition. The  $S$ -matrix  $\mathbb{S}$  is the matrix of transition elements connecting the eigenspinors. It is therefore a square matrix of dimension  $4B$ . We divide the  $S$ -matrix into  $2 \times 2$  blocks  $\mathbb{S}^{(ij)(kl)}$  defining transitions between the pair of eigenspinors

traveling from  $k$  to  $l$  and a pair traveling from  $i$  to  $j$ . The  $S$ -matrix is then defined by

$$\mathbb{S}^{(ij)(kl)} := \delta_{il} \sigma_{(ij)(ki)} u^{(ij)(ki)} e^{i\phi_{\{kl\}}} , \quad (2.1)$$

where  $u^{(ij)(ki)}$  is an element of  $SU(2)$  describing the spin transformation at the vertex  $i$  and the terms  $\sigma_{(ij)(ki)}$  define a  $2v_i \times 2v_i$  unitary matrix  $\Sigma^{(i)}$ . The Kronecker-delta in (2.1) ensures transitions only occur between bonds connected at a vertex. The phases  $\phi_{\{kl\}}$  are random variables uniformly distributed in  $[0, 2\pi]$ . They define an ensemble of matrices  $\mathbb{S}_\phi$  over which we average. Such an average is equivalent to a spectral averaging when considering the level statistics of a Dirac operator on the graph.

Time-reversal invariance requires  $\Sigma^{(i)}$  to be symmetric and

$$\mathbb{S}^{(lk)(ji)} = |\mathbb{S}^{(ij)(kl)}| (\mathbb{S}^{(ij)(kl)})^{-1} . \quad (2.2)$$

To satisfy (2.2) we define spin transformations

$$u_j^{(ij)(ki)} := u_j^{(i)} (u_k^{(i)})^{-1} . \quad (2.3)$$

The  $v_i$  elements  $u_j^{(i)} \in SU(2)$  define all spin transformations at the vertex  $i$ . See [5] for details.

Having defined the  $S$ -matrix of a Dirac graph the form factor may be introduced as in [7]. We remove Kramers' degeneracy present in systems with half-integer spin and time-reversal invariance as in [6]. This leaves us with  $N = 2B$  eigenvalues of  $\mathbb{S}$ , leading to

$$K(\tau) := \frac{1}{4N} \langle |\text{tr } \mathbb{S}_\phi^n|^2 \rangle_\phi . \quad (2.4)$$

The trace of  $\mathbb{S}^n$  may be expanded as a sum over the set  $P_n$  of periodic orbits of length  $n$ ,

$$\text{tr } \mathbb{S}_\phi^n = \sum_{p \in P_n} \frac{n}{r_p} A_p e^{i\pi\mu_p} \text{tr}(d_p) e^{i\phi_p} , \quad (2.5)$$

where the periodic orbit  $p$  consists of a series of transitions  $(b_1, b_2, \dots, b_n)$  and

$$\begin{aligned} A_p e^{i\pi\mu_p} &:= \sigma_{b_n b_{n-1}} \sigma_{b_{n-1} b_{n-2}} \dots \sigma_{b_2 b_1} , \\ d_p &:= u^{b_n b_{n-1}} u^{b_{n-1} b_{n-2}} \dots u^{b_2 b_1} , \\ \phi_p &:= \sum_{j=1}^n \phi_{\{b_j\}} . \end{aligned} \quad (2.6)$$

The phases  $\mu_p$  are such that  $A_p > 0$ , and  $r_p$  is the repetition number of  $p$  so that  $n/r_p$  is the number of possible starting positions of an orbit up to cyclic permutations. Substituting the periodic orbit expansion (2.5) into (2.4) and carrying out the average over  $\phi$  we obtain

$$K_{\text{symp}}(\tau) = \frac{n^2}{4(2B)} \sum_{p,q \in P_n} \frac{A_p A_q}{r_p r_q} e^{i\pi(\mu_p - \mu_q)} \text{tr}(d_p) \text{tr}(d_q) \delta_{\phi_p, \phi_q} \quad (2.7)$$

where  $\tau = n/2B$ . We label the form factor  $K_{\text{symp}}$  according to the symplectic symmetry introduced by time-reversal invariance in a system with spin-1/2. The Kronecker-delta fixes contributing terms in  $K_{\text{symp}}$  to pairs of orbits in which each bond is visited the same number of times. On a metric graph with rationally independent bond lengths this is equivalent to requiring the lengths of  $p$  and  $q$  be equal.

For comparison the form factor studied in [2, 1] for a graph quantised with the Schrödinger operator (spin-0) in a time-reversal symmetric fashion is

$$K_{\text{orth}}(\tau) = \frac{n^2}{2B} \sum_{p,q \in P_n} \frac{A_p A_q}{r_p r_q} e^{i\pi(\mu_p - \mu_q)} \delta_{\phi_p, \phi_q}, \quad (2.8)$$

where the definition of  $A_p$  remains the same. This form factor is labeled by the orthogonal symmetry of the system.

It was pointed out in [9] that comparing the RMT form factors of the CSE and COE makes clear the close connection between them,

$$\begin{aligned} K_{\text{CSE}}(\tau) &= \frac{\tau}{2} + \frac{\tau^2}{4} + \frac{\tau^3}{8} + \frac{\tau^4}{12} + \dots, \\ \frac{1}{2} K_{\text{COE}}\left(\frac{\tau}{2}\right) &= \frac{\tau}{2} - \frac{\tau^2}{4} + \frac{\tau^3}{8} - \frac{\tau^4}{12} + \dots. \end{aligned} \quad (2.9)$$

Calling  $K^m$  the term containing  $\tau^m$  the relationship may be written

$$K_{\text{CSE}}^m(\tau) = \left(-\frac{1}{2}\right)^{m+1} K_{\text{COE}}^m(\tau). \quad (2.10)$$

According to the conjecture of Bohigas, Giannoni, and Schmit [4] in the semiclassical limit we expect the form factors of quantum graphs to correspond to those of random matrices. In particular,

$$K_{\text{symp}}^m(\tau) = \left(-\frac{1}{2}\right)^{m+1} K_{\text{orth}}^m(\tau). \quad (2.11)$$

It is this relation we wish to demonstrate in quantum graphs.

### 3 Spin contributions to the form factor

The form factor on graphs may be studied analytically in the semiclassical limit. The system is defined by the matrix  $\mathbb{S}$  of dimension  $4B$  and so the semiclassical limit is  $B \rightarrow \infty$ . For small but finite  $\tau = n/2B$  the limit of long orbits,  $n \rightarrow \infty$ , is also required. For details see [2, 1]. In this limit the proportion of orbits  $p$  with  $r_p \neq 1$  tends to zero so these orbits can effectively be ignored in equations (2.7) and (2.8). Following [1] the sum over orbit pairs is organised in terms of diagrams. A diagram consists of all pairs of orbits related by the same pattern of permutations of arcs between self-intersections and time-reversal of arcs. Consequently such pairs of orbits have identical phases  $\phi_p = \phi_q$ . Figures 1 and 2 provide examples of diagrams.

The contribution to the form factor from a specific diagram  $D$  with  $m-1$  self-intersections is

$$K_{\text{symp}}^{m,D}(\tau) := \frac{n^2}{4(2B)} \sum_{(p,q) \in D_n} A_p A_q e^{i\pi(\mu_p - \mu_q)} \text{tr}(d_p) \text{tr}(d_q) . \quad (3.1)$$

Here  $D_n$  is the set of pairs of orbits  $(p, q)$  of length  $n$  contained in  $D$ .

To separate spin contributions from this sum we assume that the elements  $d_p$  are chosen randomly (independent of  $p$ ) from a (sub-) group  $\Gamma \subseteq \text{SU}(2)$ . This can be achieved by selecting the elements  $u_j^{(i)}$  randomly from  $\Gamma$ . Then

$$K_{\text{symp}}^{m,D}(\tau) = \frac{1}{4} \left( \frac{1}{|D_n|} \sum_{(p,q) \in D_n} \text{tr}(d_p) \text{tr}(d_q) \right) \times \left( \frac{n^2}{2B} \sum_{(p,q) \in D_n} A_p A_q e^{i\pi(\mu_p - \mu_q)} \right) . \quad (3.2)$$

The second term is the equivalent contribution to the orthogonal form factor (2.8),

$$K_{\text{orth}}^{m,D}(\tau) := \frac{n^2}{2B} \sum_{(p,q) \in D_n} A_p A_q e^{i\pi(\mu_p - \mu_q)} . \quad (3.3)$$

We will show that in the semiclassical limit

$$\langle \text{tr}(d_p) \text{tr}(d_q) \rangle^{m,D_n} := \frac{1}{|D_n|} \sum_{(p,q) \in D_n} \text{tr}(d_p) \text{tr}(d_q) \rightarrow \left( -\frac{1}{2} \right)^{m-1} , \quad (3.4)$$

independent of the diagram  $D$  for fixed  $m$ . Substituting into (3.2) generates the relation (2.11) between the orthogonal and symplectic form factors. At this point we remark that the spin contribution to the form factor defined in (3.4) is different from that in [9].

To determine the spin contributions we first take the case  $\Gamma = \text{SU}(2)$ , ie random spin rotations are chosen from the whole of  $\text{SU}(2)$  with Haar measure. Each arc of  $p$  contributes a random element of  $\text{SU}(2)$  to  $d_p$ . If  $(p, q) \in D$  in  $d_q$  the order of the product is changed and some elements are replaced with their inverse. In the semiclassical limit where the number of orbits tends to infinity the sum over pairs of orbits may be replaced by integrals over  $\text{SU}(2)$  for each arc of the diagram. To evaluate the spin contributions we require three identities:

$$\int_{\text{SU}(2)} \text{tr}(xuyu) \, du = -\frac{1}{2} \text{tr}(xy^{-1}) \quad (3.5)$$

$$\int_{\text{SU}(2)} \text{tr}(xuyu^{-1}) \, du = \frac{1}{2} \text{tr}(x) \text{tr}(y) \quad (3.6)$$

$$\int_{\text{SU}(2)} \text{tr}(xu) \text{tr}(yu) \, du = \frac{1}{2} \text{tr}(xy^{-1}) \quad (3.7)$$

where  $u, x, y \in \text{SU}(2)$ . Equations (3.5)–(3.7) may be evaluated directly by parameterising  $\text{SU}(2)$ . To establish (3.4) we consider changing the order of arcs at one self-intersection or between a pair of intersections. All diagrams can be generated via these operations. Counting the number of intersections at which the order of elements is changed then proves the result.

### 3.1 Reordering at a single intersection

Figure 1 shows two alternative orders at a single intersection. For a pair of orbits with such a self-intersection the spin contribution in the semiclassical limit is

$$\langle \text{tr}(d_p) \text{tr}(d_q) \rangle = \int_{\text{SU}(2)} \dots \int_{\text{SU}(2)} \text{tr}(\alpha \beta l_1 \gamma \delta l_2) \text{tr}(\alpha \gamma^{-1} l_3^{-1} \beta^{-1} \delta l_4) d\alpha d\beta d\gamma d\delta \dots \quad (3.8)$$

Changing variables so  $x := \alpha\beta$  and  $y := \gamma\delta$  and using (3.5) we obtain

$$\langle \text{tr}(d_p) \text{tr}(d_q) \rangle = -\frac{1}{2} \int_{\text{SU}(2)} \dots \int_{\text{SU}(2)} \text{tr}(x l_1 y l_2) \text{tr}(x l_3 y l_4) dx dy \dots \quad (3.9)$$

Reordering terms in  $d_q$  at one vertex therefore introduces a factor of  $-1/2$  to the spin contribution.

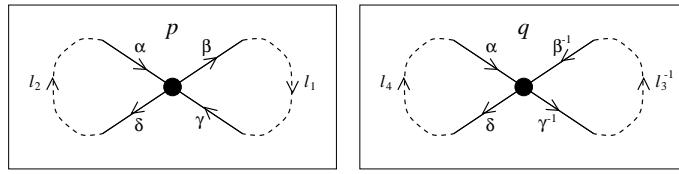


Figure 1: A pair of orbits  $(p, q)$  with the order of arcs changed at a single self-intersection

### 3.2 Reordering at a pair of intersections

The procedure in 3.1 uses the time-reversal invariance of the system to reverse the directions of arcs. There is a second type of reordering of arcs independent of this symmetry which is possible when multiple arcs run between a pair of self-intersections, figure 2. The sections  $\beta_1 \dots \alpha_2$  and  $\delta_1 \dots \gamma_2$  can then be taken in either order. The relevant spin contribution is

$$\begin{aligned} \langle \text{tr}(d_p) \text{tr}(d_q) \rangle = & \int_{\text{SU}(2)} \dots \int_{\text{SU}(2)} \text{tr}(\alpha_1 \beta_1 l_1 \alpha_2 \beta_2 l_2 \gamma_1 \delta_1 l_3 \gamma_2 \delta_2 l_4) \text{tr}(\alpha_1 \delta_1 l_7 \gamma_2 \beta_2 l_6 \gamma_1 \beta_1 l_5 \alpha_2 \delta_2 l_8) \times \\ & \times d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 d\gamma_1 d\gamma_2 d\delta_1 d\delta_2 \dots . \end{aligned} \quad (3.10)$$

We notice that exchanging the order of the central arcs changes the order of elements of  $\text{SU}(2)$  at both self-intersections. From (3.6) and (3.7) it can be shown that

$$\int_{\text{SU}(2)} \int_{\text{SU}(2)} \text{tr}(u a v^{-1} b u^{-1} c v d) du dv = \frac{1}{4} \text{tr}(c b a d) . \quad (3.11)$$

To apply this to (3.10) make substitutions  $x_j := \alpha_j \beta_j$ ,  $y_j := \gamma_j \delta_j$ , then

$$\begin{aligned} \langle \text{tr}(d_p) \text{tr}(d_q) \rangle = & \frac{1}{4} \int_{\text{SU}(2)} \dots \int_{\text{SU}(2)} \text{tr}(x_1 l_1 x_2 l_2 y_1 l_3 y_2 l_4) \text{tr}(x_1 l_5 x_2 l_6 y_1 l_7 y_2 l_8) \times \\ & \times dx_1 dx_2 dy_1 dy_2 \dots . \end{aligned} \quad (3.12)$$

Reordering by exchanging arcs of a diagram introduces a factor of  $1/4$  to the spin contribution and requires changing the order at two self-intersections simultaneously.

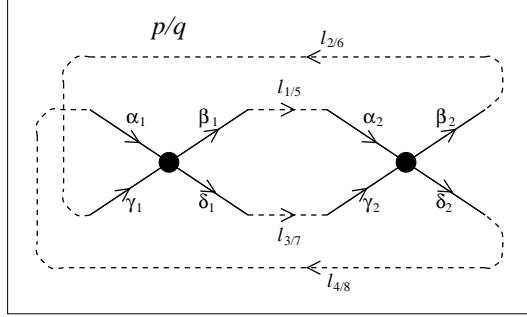


Figure 2: Reordering arcs between a pair of self-intersections

### 3.3 Counting self-intersections

All diagrams can be constructed via combinations of the procedures described in sections 3.1 and 3.2. This is not obvious, for example a system of loops in which the same self-intersection is visited twice does not appear to fall within this classification, figure 3. In fact degenerate self-intersections allow both types of reordering at the intersection. To distinguish the cases it is necessary to follow the orbit counting each intersection when it is reached after determining if the order of arcs at the intersection has been changed. The number of self-intersections for a given diagram is then  $m - 1$  (note our multiple counting of degenerate self-intersections differs from the definition in [1]). As each intersection effectively contributes a factor  $-1/2$  we obtain

$$\langle \text{tr}(d_p) \text{tr}(d_q) \rangle^{m, D_n} \rightarrow \left( -\frac{1}{2} \right)^{m-1} \int_{\text{SU}(2)} (\text{tr}(u))^2 du . \quad (3.13)$$

The final integral over  $\text{SU}(2)$  is

$$\int_{\text{SU}(2)} (\text{tr}(u))^2 du = 1 \quad (3.14)$$

as the defining representation of  $\text{SU}(2)$  is naturally irreducible.

We remark that if diagrams with  $m' - 1 \neq m - 1$  self-intersections contributed to  $K_{\text{orth}}^m(\tau)$  in such a way that nevertheless  $K_{\text{orth}}(\tau) = K_{\text{COE}}(\tau)$  the spin contribution would lead to  $K_{\text{symp}}(\tau) \neq K_{\text{CSE}}(\tau)$ . This observation supports the hypothesis that the  $\tau^m$ -term derives only from diagrams with  $m - 1$  self-intersections.

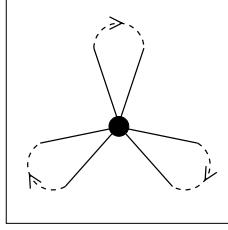


Figure 3: Orbit with degenerate self-intersection

### 3.4 Spin rotations from subgroups of $SU(2)$

If instead of the whole of  $SU(2)$  spin transformations are chosen from a subgroup  $\Gamma$  it is still possible to find the connection between CSE and COE statistics. Rather than averaging over  $SU(2)$  the identities (3.5) – (3.7) must be understood in terms of an average over  $\Gamma$ , ie

$$\int_{SU(2)} f(u) \, du \quad \text{is replaced by} \quad \frac{1}{|\Gamma|} \sum_{g \in \Gamma} f(g)$$

when  $\Gamma$  is finite. Let  $\Gamma \subset SU(2)$ , viewed as a representation, be irreducible. The identities (3.6) and (3.7) can then be derived from Schur's lemma. If and only if  $\Gamma$  is also a quaternionic representation (any representation equivalent to its complex conjugate but inequivalent to any real representation) the identity (3.5) may also be derived, see [8]. As long as (3.5) – (3.7) hold the argument is unaffected by the use of a subgroup of spin transformations. CSE statistics hence depend on the subgroup of spin transformations providing an irreducible quaternionic representation.

An example of a finite group of spin transformations are Hamilton's quaternions

$$\Gamma = \{\pm I, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\} , \quad (3.15)$$

where  $\sigma_j$  is a Pauli matrix. In [10] spin transformations from this subgroup are applied to the cat map and CSE statistics observed. As  $\Gamma$  is both irreducible and quaternionic CSE statistics can indeed be expected with spin transformations taken even from such a small subgroup of  $SU(2)$ .

The conditions (3.5) – (3.7) depend only on the representation being irreducible and quaternionic consequently the argument also generalises to higher dimensional representations of  $SU(2)$ , ie higher spins. Let  $\Gamma \subseteq SU(2)$  and  $R^s(\Gamma)$  an irreducible representation of dimension  $2s + 1$ , ie a spin  $s$  representation, then

$$\frac{1}{|\Gamma|} \sum_{g \in \Gamma} \text{tr} \left( X (R^s(g))^2 \right) = \frac{c}{2s + 1} \text{tr}(X) . \quad (3.16)$$

Here  $c = 1$  for real representations and  $c = -1$  for quaternionic representations, however only quaternionic representations can have even dimension [8]. Therefore  $c = -1$ , compare (3.5), implies  $s$  half-integer. (3.6) and (3.7) generalise similarly. The RMT relation between

the symplectic and orthogonal form factors (2.11) can hence be derived for half-integer spin provided spin-transitions generate a quaternionic irreducible representation of  $\Gamma$ .

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## References

- [1] G. Berkolaiko, H. Schanz, and R. S. Whitney. Form factor for a family of quantum graphs: An expansion to third order. *preprint arXiv:nlin.CD/0205014*, 2002.
- [2] G. Berkolaiko, H. Schanz, and R. S. Whitney. Leading off-diagonal correction to the form factor of large graphs. *Phys. Rev. Lett.*, 82:104101, 2002.
- [3] M. V. Berry. Semiclassical theory of spectral rigidity. *Proc. Roy. Soc. Lond. A*, 400:229–251, 1985.
- [4] O. Bohigas, M.-J. Giannoni, and C. Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. *Phys. Rev. Lett.*, 52:1–4, 1984.
- [5] J. Bolte and J. M. Harrison. Spectral statistics for the Dirac operator on graphs. *J. Phys. A: Math. Gen.*, 36:2747–2769, 2003.
- [6] J. Bolte and S. Keppeler. Semiclassical form factor for chaotic systems with spin  $\frac{1}{2}$ . *J. Phys. A: Math. Gen.*, 32:8863–8880, 1999.
- [7] F. Haake. *Quantum Signatures of Chaos*. Springer, 2nd edition, 2001.
- [8] M. Hamermesh. *Group Theory and its Application to Physical Problems*. Addison-Wesley, 1962.
- [9] S. Heusler. The semiclassical origin of the logarithmic singularity in the symplectic form factor. *J. Phys. A: Math. Gen.*, 34:L483–L490, 2001.
- [10] S. Keppeler, J. Marklof, and F. Mezzadri. Quantum cat maps with spin 1/2. *Nonlinearity*, 14:719–738, 2001.
- [11] M. Sieber. Leading off-diagonal approximation for the spectral form factor for uniformly hyperbolic systems. *J. Phys. A: Math. Gen.*, 35:L613–L619, 2002.
- [12] M. Sieber and K. Richter. Correlations between periodic orbits and their rôle in spectral statistics. *Physica Scripta*, T90:128, 2001.